SOME SHARP WILKER TYPE INEQUALITIES AND THEIR APPLICATIONS

ZHENG-HANG YANG

ABSTRACT. In this paper, we prove that for fixed $k \geq 1$, the Wilker type inequality

$$\frac{2}{k+2}\left(\frac{\sin x}{x}\right)^{kp} + \frac{k}{k+2}\left(\frac{\tan x}{x}\right)^{p} > 1$$

holds for $x \in (0, \pi/2)$ if and only if $p > 0$ or $p \leq -\frac{\ln(k+2)-\ln 2}{k(\ln \pi - \ln 2)}$. It is reversed if and only if $-\frac{12}{5(k+2)} \leq p < 0$. Its hyperbolic version holds for $x \in (0, \infty)$ if and only if $p > 0$ or $p \leq -\frac{12}{5(k+2)}$. And, for fixed $k < -2$, the hyperbolic version is reversed if and only if $p < 0$ or $p \geq -\frac{12}{5(k+2)}$. Our results unify and generalize some known ones.

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1. INTRODUCTION

Wilker [18] proposed two open problems, the first of which states that if $x \in (0, \pi/2)$ then

$$(1.1) \qquad \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2,$$

which was proved by Sumner et al. in [17].

Wilker inequality (1.1) and the second one have attracted great interest of many mathematicians and have produced a batch of Wilker type ones by various generalizing and improving as well as different methods and ideas (see [1], [2], [3], [9], [11], [12], [21], [22], [19], [20], [24], [25], [28], [26], [29] and related references therein).

In [19], Wu and Srivastava established another Wilker type inequality

$$(1.2) \qquad \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2 \ \text{ for } x \in (0, \pi/2),$$

and proved a weighted and exponential generalization of Wilker inequality.

Theorem Wu ([19, Theorem 1]). *Let $\lambda > 0, \mu > 0$ and $p \leq 2q\mu/\lambda$. If $q > 0$ or $q \leq \min(-1, -\lambda/\mu)$, then*

$$(1.3) \qquad \frac{\lambda}{\lambda + \mu}\left(\frac{\sin x}{x}\right)^p + \frac{\mu}{\lambda + \mu}\left(\frac{\tan x}{x}\right)^q > 1$$

holds for $x \in (0, \pi/2)$.

As an application of the inequality (1.3), an open problem posed by the Sádor–Bencze in [13] was solved and improved. Recently, the inequality (1.3) and all

results in [19] were extended in [1] to Bessel functions. A hyperbolic version of
Theorem Wu has been presented in [22] very recently.

In 2009, Zhu [28] gave another exponential generalization of Wilker inequality
(1.1) as follows.

Theorem Zh1 ([28, Theorem 1.1, 1.2]). *Let $0 < x < \pi/2$. Then the inequalities*

$$(1.4) \qquad \left(\frac{\sin x}{x}\right)^{2p} + \left(\frac{\tan x}{x}\right)^{p} > \left(\frac{x}{\sin x}\right)^{2p} + \left(\frac{x}{\tan x}\right)^{p} > 2$$

hold if $p \geq 1$, while the first one in (1.4) holds if and only if $p > 0$.

Theorem Zh2 ([28, Theorem 1.3, 1.4]). *Let $x > 0$. Then the inequalities*

$$(1.5) \qquad \left(\frac{\sinh x}{x}\right)^{2p} + \left(\frac{\tanh x}{x}\right)^{p} > \left(\frac{x}{\sinh x}\right)^{2p} + \left(\frac{x}{\tanh x}\right)^{p} > 2$$

hold if $p \geq 1$, while the first one in (1.5) holds if and only if $p > 0$.

In the end of the same paper, Zhu posed two open problems: find the respective
largest range of p such that the inequalities (1.4) and (1.5) hold. They have been
solved by Matejička in [7].

Another inequality associated with Wilker one is the following

$$(1.6) \qquad\qquad 2\frac{\sin x}{x} + \frac{\tan x}{x} > 3$$

for $x \in (0, \pi/2)$, which is known as Huygens inequality [4]. The following refinement
of Huyegens inequality is due to Neuman and Sándor [11]:

$$(1.7) \qquad\qquad 2\frac{\sin x}{x} + \frac{\tan x}{x} > 2\frac{x}{\sin x} + \frac{x}{\tan x} > 3,$$

where $x \in (0, \pi/2)$. Very recently, the generalizations of (1.7), similar to (1.4), has
been derived by Neuman in [11]. In [27], Zhu proved that for $x \in (0, \pi/2)$

$$(1.8) \qquad (1 - \xi_1)\frac{\sin x}{x} + \xi_1\frac{\tan x}{x} \quad > \quad 1 > (1 - \eta_1)\frac{\sin x}{x} + \eta_1\frac{\tan x}{x},$$

$$(1.9) \qquad (1 - \xi_2)\frac{x}{\sin x} + \xi_2\frac{x}{\tan x} \quad > \quad 1 > (1 - \eta_2)\frac{x}{\sin x} + \eta_2\frac{x}{\tan x}$$

with the best constants $\xi_1 = 1/3$, $\eta_1 = 0$, $\xi_2 = 1/3$, $\eta_2 = 1 - 2/\pi$. Later, he in
[26] generalized inequalities (1.8) and (1.9) in exponential form, which is stated as
follows.

Theorem Zh3 ([28, Theorem 1.1, 1.2]). *Let $0 < x < \pi/2$. Then we have*

    (i) when $p \geq 1$, the double inequality

$$(1.10) \qquad (1 - \lambda)\left(\frac{x}{\sin x}\right)^{p} + \lambda\left(\frac{x}{\tan x}\right)^{p} < 1 < (1 - \eta)\left(\frac{x}{\sin x}\right)^{p} + \eta\left(\frac{x}{\tan x}\right)^{p}$$

holds if and only if $\eta \leq 1/3$ and $\lambda \geq 1 - (2/\pi)^{p}$.

    (ii) when $0 \leq p \leq 4/5$, the double inequality (1.10) holds if and only if $\lambda \geq 1/3$
and $\eta \leq 1 - (2/\pi)^{p}$.

    (iii) when $p < 0$, the second one in (1.10) holds if and only if $\eta \geq 1/3$.

The hyperbolic version of inequalities (1.7) was given in [11] by Neuman and
Sándor. In the same year, Zhu showed that

Theorem Zh4 ([29, Theorem 4.1]). *Let $x > 0$. Then*

    (i) when $p \geq 4/5$, the double inequality

$$(1.11) \quad (1 - \lambda)\left(\frac{x}{\sinh x}\right)^{p} + \lambda\left(\frac{x}{\tanh x}\right)^{p} < 1 < (1 - \eta)\left(\frac{x}{\sinh x}\right)^{p} + \eta\left(\frac{x}{\tanh x}\right)^{p}$$

holds if and only if $\eta \geq 1/3$ and $\lambda \leq 0$;
    (ii) when $p < 0$, the inequality

$$(1.12) \qquad (1 - \eta) \left(\frac{x}{\sinh x}\right)^p + \eta \left(\frac{x}{\tanh x}\right)^p > 1$$

holds if and only if $\eta \leq 1/3$.
    The aim of this paper is to find the best p such that the inequalities

$$(1.13) \qquad \frac{2}{k+2} \left(\frac{\sin x}{x}\right)^{kp} + \frac{k}{k+2} \left(\frac{\tan x}{x}\right)^p \; > \; 1 \text{ for } x \in (0, \pi/2),$$

$$(1.14) \qquad \frac{2}{k+2} \left(\frac{\sinh x}{x}\right)^{kp} + \frac{k}{k+2} \left(\frac{\tanh x}{x}\right)^p \; > \; 1 \text{ for } x \in (0, \infty)$$

or their reverse ones hold for certain fixed k with $k(k+2) \neq 0$. In Section 2, some useful lemmas are proved. necessary and sufficient conditions for (1.13) or its reverse and (1.14) to hold are presented in Section 3. Some applications of our main results given in Section 4.

2. LEMMAS

The following two lemmas is very important in the sequel.

Lemma 1. *Let A, B and C be defined on $(0, \pi/2)$ by*

$$(2.1) \qquad A \;=\; A(x) = (\cos x)(\sin x - x \cos x)^2 (x - \cos x \sin x),$$

$$(2.2) \qquad B \;=\; B(x) = (x - \cos x \sin x)^2 (\sin x - x \cos x),$$

$$(2.3) \qquad C \;=\; C(x) = x \left(\sin^2 x\right) \left(-2x^2 \cos x + x \sin x + \cos x \sin^2 x\right).$$

Then for fixed $k \geq 1$ the $x \mapsto C(x) / (kA(x) + B(x))$ is increasing on $(0, \pi/2)$. Moreover, we have

$$(2.4) \qquad \tfrac{5}{12(k+2)} < \frac{C(x)}{kA(x) + B(x)} < 1.$$

Proof. Evidently, A, $B > 0$ for $x \in (0, \pi/2)$ due to $(\sin x - x \cos x) > 0$ and $(x - \cos x \sin x) = (2x - \sin 2x)/2 > 0$, while $C > 0$ because

$$\left(-2x^2 \cos x + x \sin x + \cos x \sin^2 x\right) = x^2 (\cos x) \left(\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right) > 0$$

by Wilker inequality (1.1).
    Denote $(kA + B)/C$ by D and factoring yields

$$
\begin{aligned}
D(x) \;&=\; \frac{x\left(\sin^2 x\right)\left(-2x^2 \cos x + x \sin x + \cos x \sin^2 x\right)}{(\sin x - x \cos x)(x - \cos x \sin x)((1 - k \cos^2 x)x + (k-1)\cos x \sin x)} \\
&=\; \frac{-2x^2 \cos x + x \sin x + \cos x \sin^2 x}{(\sin x - x \cos x)(x - \cos x \sin x)} \times \frac{x \sin^2 x}{k(\sin x - x \cos x)\cos x + (x - \cos x \sin x)} \\
&: \;=\; D_1(x) \times D_2(x).
\end{aligned}
$$

It is known that the function D_1 (which is equal to G in [28, Proof of Lemma 2.9]) is positive and increasing on $(0, \pi/2)$ proved in [28, Proof of Lemma 2.9], and it remains to prove the function D_2 is also positive and increasing. Clearly, $D_2(x) > 0$,

we only need to show that $D_2'(x) > 0$ for $x \in (0, \pi/2)$. Indeed, differentiation and simplifying yield

$$
\begin{aligned}
D_2'(x) &= (k-1)(\sin x)\frac{\left(-2x^2\cos x + \cos x\sin^2 x + x\sin x\right)}{\left(k(\sin x - x\cos x)\cos x + (x - \cos x\sin x)\right)^2} \\
&= \frac{(k-1)x^2\sin x\cos x}{\left(k(\sin x - x\cos x)\cos x + (x - \cos x\sin x)\right)^2}\left(\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right),
\end{aligned}
$$

which is clearly positive due to Wilker inequality (1.1). Hence, $C/(kA + B)$ is increasing on $(0, \pi/2)$, and it is deduced that

$$
\tfrac{5}{12(k+2)} = \lim_{x\to 0}\frac{C(x)}{kA(x) + B(x)} < D(x) < \lim_{x\to\pi/2-}\frac{C(x)}{kA(x) + B(x)} = 1.
$$

This completes the proof. $\qquad\square$

Lemma 2. *Let U, V and W be defined on $(0, \infty)$ by*

$$(2.5)\quad E = E(x) = (\cosh x)(\sinh x - x\cosh x)^2(x - \cosh x\sinh x),$$

$$(2.6)\quad F = F(x) = (\sinh x - x\cosh x)(x - \cosh x\sinh x)^2,$$

$$(2.7)\quad G = G(x) = x\left(\sinh^2 x\right)\left(2x^2\cosh x - x\sinh x - \cosh x\sinh^2 x\right).$$

Then for fixed $k \geq 1$ (or $k < -2$) the function $x \mapsto G(x)/(kE(x) + F(x))$ is decreasing (increasing) on $(0, \infty)$. Moreover, we have

$$(2.8)\quad \min\left(0, \frac{12}{5(k+2)}\right) < \frac{G(x)}{kE(x) + F(x)} < \max\left(0, \frac{12}{5(k+2)}\right).$$

Proof. It is easy to verify that E, $F < 0$ for $x \in (0, \infty)$ due to

$$
\begin{aligned}
(x - \cosh x\sinh x) &= (2x - \sinh 2x)/2 < 0, \\
(\sinh x - x\cosh x) &= x\left(\frac{\sinh x}{x} - \cos x\right) < 0.
\end{aligned}
$$

While $G < 0$ because

$$
\left(2x^2\cosh x - x\sinh x - \cosh x\sinh^2 x\right) = -x^2(\cosh x)\left(\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} - 2\right) < 0
$$

by Wilker inequality (1.5).

Denote $G/(kE + F)$ by H and factoring give

$$
\begin{aligned}
H(x) &= \frac{x\left(\sinh^2 x\right)\left(2x^2\cosh x - x\sinh x - \cosh x\sinh^2 x\right)}{(\cosh x)(\sinh x - x\cosh x)^2(x - \sinh x\cosh x)k + (\sinh x - x\cosh x)(x - \sinh x\cosh x)^2} \\
&= \frac{-2x^2\cosh x + x\sinh x + \cosh x\sinh^2 x}{(x\cosh x - \sinh x)(\sinh x\cosh x - x)} \times \frac{x\left(\sinh^2 x\right)}{(k(x\cosh x - \sinh x)\cosh x + \sinh x\cosh x - x)} \\
&: = H_1(x) \times H_2(x).
\end{aligned}
$$

Clearly, $H_1(x) > 0$, and it has shown in [7, Proof of Lemma 2.2] that H_1 (that is, the function s, in [7, Proof of Lemma 2.2]) is decreasing on $(0, \infty)$. In order to prove the monotonicity of H, we also need to deal with the sign and monotonicity of H_2.

(i) Clearly, $H_2(x) > 0$ for $k \geq 1$. And, we claim that H_2 is also decreasing on $(0, \infty)$. Indeed, differentiation and simplifying yield

$$
\begin{aligned}
H_2'(x) &= -(k-1)\sinh x \frac{\left(-2x^2\cosh x + \cosh x \sinh^2 x + x\sinh x\right)}{(x\cosh x - \sinh x)^2 (\cosh x \sinh x - x)^2} \\
&= -\frac{(k-1)x^2 \sinh x \cosh x}{(x\cosh x - \sinh x)^2 (\cosh x \sinh x - x)^2}\left(\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} - 2\right) < 0.
\end{aligned}
$$

Consequently, $H = H_1 \times H_2$ is positive and decreasing on $(0, \infty)$, and so

$$
0 = \lim_{x\to\infty}\frac{G(x)}{kE(x) + F(x)} < \frac{G(x)}{kE(x) + F(x)} < \lim_{x\to 0}\frac{G(x)}{kE(x) + F(x)} = \frac{12}{5(k+2)}.
$$

(ii) For $k < -2$, by the previous proof we see that $-H_2'$ is decreasing on $(0, \infty)$, and so

$$
0 < -\frac{1}{k} = \lim_{x\to\infty}(-H_2(x)) < -H_2(x) < \lim_{x\to 0}(-H_2(x)) = -\frac{3}{k+2}.
$$

It is implied that $-H_2$ is positive and decreasing on $(0, \infty)$, and so is the function $-H = H_1 \times (-H_2)$. That is, H is negative and increasing on $(0, \infty)$, and (2.8) naturally holds.

This completes the proof. $\qquad\square$

Remark 1. *It should be noted that $kE(x) + F(x) < 0$ for $k \geq 1$ and $kE(x) + F(x) > 0$ for $k < -2$. In fact, it suffices to notice (2.8) and $G(x) < 0$ for $x \in (0, \infty)$.*

Lemma 3. *For $k \geq 1$, we have*

$$
1 > \frac{\ln(k+2) - \ln 2}{k(\ln\pi - \ln 2)} > \frac{12}{5(k+2)}.
$$

Proof. It suffices to show that

$$
\begin{aligned}
\delta_1(k) &= \frac{\ln(k+2) - \ln 2}{\ln\pi - \ln 2} - k < 0, \\
\delta_2(k) &= \frac{\ln(k+2) - \ln 2}{\ln\pi - \ln 2} - \frac{12k}{5(k+2)} > 0
\end{aligned}
$$

where $k \geq 1$.

Differentiation gives

$$
\begin{aligned}
\delta_1'(k) &= \frac{1}{(\ln\pi - \ln 2)(k+2)} - 1 < 0, \\
\delta_2'(k) &= \frac{1}{5}\frac{5k + 24\ln 2 - 24\ln\pi + 10}{(k+2)^2(\ln\pi - \ln 2)} > 0
\end{aligned}
$$

for $k \geq 1$. It follows that $\delta_1(k) \leq \delta_1(1) = (\ln 3 - \ln 2)/(\ln 3 - \ln\pi) < 0$, $\delta(k) \geq \delta(1) = (\ln 3 - \ln 2)/(\ln\pi - \ln 2) - 4/5 > 0$, which proves the lemma. $\qquad\square$

3. MAIN RESULTS

Theorem 1. *For fixed $k \geq 1$, the inequality (1.13) holds for $x \in (0, \pi/2)$ if and only if $p > 0$ or $p \leq -\frac{\ln(k+2) - \ln 2}{k(\ln\pi - \ln 2)}$*

Proof. The inequality (1.13) is equivalent to

$$(3.1) \qquad f(x) = \frac{2}{k+2}\left(\frac{\sin x}{x}\right)^{kp} + \frac{k}{k+2}\left(\frac{\tan x}{x}\right)^{p} - 1 > 0$$

for $x \in (0, \pi/2)$. Differentiation yields

$$f'(x) = -\frac{2kp}{k+2}\frac{\sin x - x\cos x}{x^2}\left(\frac{\sin x}{x}\right)^{kp-1} + \frac{kp}{k+2}\frac{x - \sin x\cos x}{x^2\cos^2 x}\left(\frac{\tan x}{x}\right)^{p-1}$$

$$(3.2) \qquad = \frac{kp}{k+2}\frac{x - \sin x\cos x}{x^2\cos^2 x}\left(\frac{\tan x}{x}\right)^{p-1}g(x),$$

where

$$(3.3) \qquad g(x) = 1 - 4\frac{\sin x - x\cos x}{2x - \sin 2x}\left(\frac{\sin x}{x}\right)^{(k-1)p}(\cos x)^{p+1}.$$

Simple computation leads to $g(0^+) = 0$.

Differentiation again and simplifying give

$$(3.4) \qquad g'(x) = 8\frac{\left(\frac{\sin x}{x}\right)^{(k-1)p}(\cos x)^p}{x(\sin x)(2x - \sin 2x)^2}h(x),$$

where

$$h(x) = (\cos x)(\sin x - x\cos x)^2(x - \cos x\sin x)kp$$
$$+ (x - \cos x\sin x)^2(\sin x - x\cos x)p$$
$$+ x(\sin^2 x)\left(-2x^2\cos x + x\sin x + \cos x\sin^2 x\right)$$
$$(3.5) \qquad = kpA(x) + pB(x) + C(x)$$
$$= (kA + B)\left(p + \frac{C}{kA + B}\right),$$

here $A(x), B(x), C(x)$ are defined by (2.1), (2.2), (2.3), respectively.

By (3.2), (3.4) we easily get

$$(3.6) \qquad \operatorname{sgn} f'(x) = \operatorname{sgn} p \operatorname{sgn} g(x),$$
$$(3.7) \qquad \operatorname{sgn} g'(x) = \operatorname{sgn} h(x).$$

Necessity. We first present two limit relations:

$$(3.8) \qquad \lim_{x \to 0^+} x^4 f(x) = \frac{kp}{36}\left(p + \frac{12}{5(k+2)}\right),$$

$$(3.9) \qquad \lim_{x \to (\pi/2)^-} f(x) = \begin{cases} \infty & \text{if } p > 0, \\ \frac{2}{k+2}\left(\frac{2}{\pi}\right)^{kp} - 1 & \text{if } p < 0. \end{cases}$$

In fact, using power series extension yields

$$f(x) = \frac{kp}{36}\frac{kp + 2p + 12/5}{k+2}x^4 + O\left(x^6\right),$$

which implies the first limit relation (3.8). From the fact $\lim_{x \to \pi/2^-} \tan x = \infty$ the second one (3.9) easily follows.

Now we can derive the necessary condition for (1.13) to holds for $x \in (0, \pi/2)$ from the simultaneous inequalities $\lim_{x \to 0^+} x^4 f(x) \geq 0$ and $\lim_{x \to (\pi/2)^-} f(x) \geq 0$. Solving for p yields $p > 0$ or

$$p \leq \min\left(-\frac{12}{5(k+2)}, -\frac{\ln(k+2) - \ln 2}{k(\ln \pi - \ln 2)}\right) = -\frac{\ln(k+2) - \ln 2}{k(\ln \pi - \ln 2)},$$

where the equality holds is due to the Lemma 3.

Sufficiency. We prove the condition $p > 0$ or $p \leq -\frac{\ln(k+2) - \ln 2}{k(\ln \pi - \ln 2)}$ is sufficient. We distinguish three cases.

Case 1: $p > 0$. Clearly, $h(x) > 0$, then $g'(x) > 0$, and then $g(x) > g(0^+) = 0$, which together with $\operatorname{sgn} p = 1$ yields $f'(x) > 0$. Then $f(x) > f(0^+) = 0$.

Case 2: $p \leq -1$. By Lemma (1) it is easy to get

$$p + \frac{C}{kA + B} < p + 1 \leq 0,$$

which reveals that $h(x) < 0$, then $g'(x) < 0$, and then $g(x) < g(0^+) = 0$, which in combination with $\operatorname{sgn} p = -1$ implies $f'(x) > 0$. Then $f(x) > f(0^+) = 0$.

Case 3: $-1 < p \leq -\frac{\ln(k+2) - \ln 2}{k(\ln \pi - \ln 2)}$. Lemma (1) reveals that $\frac{C}{kA + B}$ is increasing on $(0, \pi/2)$, so is the function $x \mapsto p + \frac{C}{kA + B} := \lambda(x)$. Since

$$\lambda(0^+) = p + \frac{12}{5(k+2)} < 0, \quad \lambda\left(\frac{\pi}{2}^-\right) = p + 1 > 0,$$

there is a unique $x_1 \in (0, \pi/2)$ such that $\lambda(x) < 0$ for $x \in (0, x_1)$ and $\lambda(x) > 0$ for $x \in (x_1, \pi/2)$, and so is $g'(x)$. Therefore, $g(x) < g(0^+) = 0$ for $x \in (0, x_1)$ but $g(\pi/2^-) = 1$, which implies that there is a sole $x_0 \in (x_1, \pi/2)$ such that $g(x) < 0$ for $x \in (0, x_0)$ and $g(x) > 0$ for $x \in (x_0, \pi/2)$. Due to $\operatorname{sgn} p = -1$ it is deduced that $f'(x) > 0$ for $x \in (0, x_0)$ and $f'(x) < 0$ for $x \in (x_0, \pi/2)$, which reveals that f is increasing on $(0, x_0)$ and decreasing on $(x_0, \pi/2)$. It follows that

$$0 \quad = \quad f(0^+) < f(x) < f(x_0) = 0 \text{ for } x \in (0, x_0),$$

$$f(x_0) \quad > \quad f(x) > f(\pi/2^-) = \frac{2}{k+2}\left(\frac{2}{\pi}\right)^{kp} - 1 \geq 0 \text{ for } x \in (x_0, \pi/2),$$

that is, $f(x) > 0$ for $x \in (0, \pi/2)$.

This completes the proof. $\qquad \square$

Theorem 2. *For fixed $k \geq 1$, the reverse of (1.13), that is,*

$$(3.10) \qquad \frac{2}{k+2}\left(\frac{\sin x}{x}\right)^{kp} + \frac{k}{k+2}\left(\frac{\tan x}{x}\right)^p < 1$$

holds for $x \in (0, \pi/2)$ if and only if $-\frac{12}{5(k+2)} \leq p < 0$.

Proof. **Necessity**. If inequality (3.10) holds for $x \in (0, \pi/2)$, then we have

$$\lim_{x \to 0^+} \frac{f(x)}{x^4} = \frac{kp}{36}\left(p + \frac{12}{5(k+2)}\right) \leq 0.$$

Solving the inequalities for p yields $-\frac{12}{5(k+2)} \leq p < 0$.

Sufficiency. We prove the condition $-\frac{12}{5(k+2)} \leq p < 0$ is sufficient. It suffices to show that $f(x) < 0$ for $x \in (0, \pi/2)$. By Lemma (1) it is easy to get

$$p + \frac{C}{kA + B} \geq p + \frac{12}{5(k+2)} \geq 0,$$

which reveals that $h(x) > 0$, then $g'(x) > 0$, and then $g(x) > g(0^+) = 0$. It in combination with $\operatorname{sgn} p = -1$ implies $f'(x) < 0$. Thus, $f(x) < f(0^+) = 0$, which proves the sufficiency and the proof is complete. $\qquad\qquad\square$

Theorem 3. *For fixed $k \geq 1$, the inequality (1.14) holds for $x \in (0, \infty)$ if and only if $p > 0$ or $p \leq -\frac{12}{5(k+2)}$.*

Proof. We define

$$(3.11) \qquad u(x) = \frac{2}{k+2}\left(\frac{\sinh x}{x}\right)^{kp} + \frac{k}{k+2}\left(\frac{\tanh x}{x}\right)^{p} - 1.$$

Then inequality (1.14) is equivalent to $u(x) > 0$. Differentiation leads to

$$(3.12) \qquad u'(x) = -\frac{kp}{2(k+2)}\frac{\sinh 2x - 2x}{x^2 \cosh^2 x}\left(\frac{\tanh x}{x}\right)^{p-1} v(x),$$

where

$$(3.13) \qquad v(x) = 1 - 4\frac{\sinh x - x\cosh x}{2x - \sinh 2x}\left(\frac{\sinh x}{x}\right)^{kp-p}(\cosh x)^{p+1}.$$

Differentiation again gives

$$(3.14) \qquad v'(x) = \frac{2(\cosh^p x)\left(\frac{\sinh x}{x}\right)^{kp-p}}{(x\sinh x)(x - \cosh x \sinh x)^2}w(x),$$

where

$$
\begin{aligned}
w(x) &= (\cosh x)(\sinh x - x\cosh x)^2(x - \cosh x \sinh x)kp \\
&\quad + (\sinh x - x\cosh x)(x - \cosh x \sinh x)^2 p \\
&\quad + x\left(\sinh^2 x\right)\left(2x^2\cosh x - x\sinh x - \cosh x \sinh^2 x\right) \\
(3.15) \qquad &= kpE(x) + pF(x) + G(x) = (kE + F)\left(p + \frac{G}{kE + F}\right),
\end{aligned}
$$

here $E(x), F(x), G(x)$ are defined by (2.5), (2.6), (2.7), respectively.

By (3.12), (3.14) we easily get

$$(3.16) \qquad \operatorname{sgn} u'(x) = -\operatorname{sgn}\frac{k}{k+2}\operatorname{sgn} p\operatorname{sgn} v(x),$$

$$(3.17) \qquad \operatorname{sgn} v'(x) = \operatorname{sgn} w(x).$$

Necessity. If inequality (1.14) holds for $x \in (0, \infty)$, then we have $\lim_{x\to 0^+} x^{-4}u(x) \geq 0$. Expanding $u(x)$ in power series gives

$$u(x) = \frac{k}{36}p\left(p + \frac{12}{5p(k+2)}\right)x^4 + O\left(x^6\right).$$

Hence we get

$$\lim_{x\to 0^+} x^{-4}u(x) = \frac{k}{36}p\left(p + \frac{12}{5(k+2)}\right) \geq 0.$$

Solving the inequality for p yields $p > 0$ or $p \leq -\frac{12}{5(k+2)}$.

Sufficiency. We prove the condition $p > 0$ or $p \leq -\frac{12}{5(k+2)}$ is sufficient for (1.14) to hold.

If $p > 0$, then $w(x) < 0$ due to $E, F, G < 0$. Hence, from (3.17) we have $v'(x) < 0$, and then $v(x) < \lim_{x \to 0^+} v(x) = 0$. It is derived by (3.16) that $u'(x) > 0$, and so $u(x) > \lim_{x \to 0^+} u(x) = 0$.

If $p \le -\frac{12}{5(k+2)}$, then by Lemma 2 we have

$$p + \frac{G}{kE + F} \le -\frac{12}{5(k+2)} + \frac{G}{kE + F} < 0,$$

and then

$$w(x) = (kE + F)\left(p + \frac{G}{kE + F}\right) > 0.$$

From (3.17) we have $v'(x) > 0$, and then $v(x) > \lim_{x \to 0^+} v(x) = 0$. It follows by (3.16) that $u'(x) > 0$, which implies that $u(x) > \lim_{x \to 0^+} u(x) = 0$.

This completes the proof. $\square$

Remark 2. *For $k \ge 1$, since $\lim_{x \to \infty} u(x) = \infty$ for $p \ne 0$ and $\lim_{x \to \infty} u(x) = 0$ for $p = 0$, there has no p such that the reverse inequality of (1.14) holds for all $x > 0$. But we can show that there is a unique $x_0 \in (0, \infty)$ such that $u(x) < 0$, that is, the reverse inequality of (1.14), for $-\frac{12}{5(k+2)} < p < 0$. The details of proof are omitted.*

Theorem 4. *For fixed $k < -2$, the reverse of (1.14), that is,*

$$(3.18) \qquad \frac{2}{k+2}\left(\frac{\sinh x}{x}\right)^{kp} + \frac{k}{k+2}\left(\frac{\tanh x}{x}\right)^{p} < 1$$

holds for $x \in (0, \infty)$ if and only if $p < 0$ or $p \ge -\frac{12}{5(k+2)}$.

Proof. **Necessity.** If inequality (1.14) holds for $x \in (0, \infty)$, then we have

$$\lim_{x \to 0^+} \frac{u(x)}{x^4} = \frac{k}{36}p\left(p + \frac{12}{5(k+2)}\right) \le 0.$$

Solving the inequality for p yields $p < 0$ or $p \ge -\frac{12}{5(k+2)}$.

Sufficiency. We prove the condition $p < 0$ or $p \ge -\frac{12}{5(k+2)}$ is sufficient for (1.14) to hold.

If $p < 0$, then $w(x) = (kE + F)\left(p + \frac{G}{kE+F}\right) < 0$ due to $kE + F > 0$ and $G < 0$. Hence, from (3.17) we have $v'(x) < 0$, and then $v(x) < \lim_{x \to 0^+} v(x) = 0$. It is derived by (3.16) that $u'(x) < 0$, and so $u(x) < \lim_{x \to 0^+} u(x) = 0$.

If $p \ge -\frac{12}{5(k+2)}$, then by Lemma 2 we have

$$p + \frac{G}{kE + F} \ge p + \frac{12}{5(k+2)} > 0,$$

and then

$$w(x) = (kE + F)\left(p + \frac{G}{kE + F}\right) > 0.$$

From (3.17) we have $v'(x) > 0$, and then $v(x) > \lim_{x \to 0^+} v(x) = 0$. It follows by (3.16) that $u'(x) < 0$, which implies that $u(x) < \lim_{x \to 0^+} u(x) = 0$.

This completes the proof. $\square$

4. Applications

4.1. **Huygens type inequalities.** Letting $k = 1$ in Theorem 1 and 2, we have

Proposition 1. *For $x \in (0, \pi/2)$, inequality*

$$(4.1) \qquad \frac{2}{3} \left(\frac{\sin x}{x} \right)^p + \frac{1}{3} \left(\frac{\tan x}{x} \right)^p > 1 > \frac{2}{3} \left(\frac{\sin x}{x} \right)^q + \frac{1}{3} \left(\frac{\tan x}{x} \right)^q$$

holds if and only if $p > 0$ or $p \leq -\frac{\ln 3 - \ln 2}{\ln \pi - \ln 2} \approx -0.898$ and $-4/5 \leq q < 0$.

Let $M_r(a, b; w)$ denote the r-th weighted power mean of positive numbers $a, b > 0$ defined by

$$(4.2) \quad M_r(a, b; w) := (wa^r + (1 - w) b^r)^{1/r} \text{ if } r \neq 0 \text{ and } M_0(a, b; w) = a^w b^{1-w},$$

where $w \in (0, 1)$.

Since

$$\frac{2}{3} \left(\frac{\sin x}{x} \right)^p + \frac{1}{3} \left(\frac{\tan x}{x} \right)^p = \frac{\frac{2}{3} + \frac{1}{3} (\cos x)^{-p}}{\left(\frac{\sin x}{x} \right)^{-p}},$$

by Proposition 1 the inequality

$$\frac{\sin x}{x} > \left(\frac{2}{3} + \frac{1}{3} (\cos x)^{-p} \right)^{-1/p} = M_{-p} \left(1, \cos x; \tfrac{2}{3} \right)$$

holds for $x \in (0, \pi/2)$ if and only if $-p \leq 4/5$. Similarly, its reverse one holds if and only if $-p \geq \frac{\ln 3 - \ln 2}{\ln \pi - \ln 2}$. The facts cab be stated as a corollary.

Corollary 1. *Let $M_r(a, b; w)$ be defined by (4.2). Then for $x \in (0, \pi/2)$, the inequalities*

$$(4.3) \qquad M_\alpha \left(1, \cos x; \tfrac{2}{3} \right) < \frac{\sin x}{x} < M_\beta \left(1, \cos x; \tfrac{2}{3} \right)$$

hold if and only if $\alpha \leq 4/5$ and $\beta \geq \frac{\ln 3 - \ln 2}{\ln \pi - \ln 2} \approx -0.898$.

Remark 3. *Cusa-Huygens inequality [4] refers to*

$$(4.4) \qquad \frac{\sin x}{x} < \tfrac{2}{3} + \tfrac{1}{3} \cos x$$

holds for $x \in (0, \pi/2)$, which is an equivalent one of the second one in (1.7). As an improvement and generalization, Corollary 1 was proved in [23] by Yang. Here we provide a new proof.

Remark 4. *Let $a > b > 0$ and let $x = \arcsin \frac{a-b}{a+b} \in (0, \pi/2)$. Then $(\sin x)/x = P/A$, $\cos x = G/A$, and then inequalities (4.3) can be changed into*

$$(4.5) \qquad M_\alpha \left(A, G; \tfrac{2}{3} \right) < P < M_\beta \left(A, G; \tfrac{2}{3} \right),$$

where P is the first Seiffert mean [14] defined by

$$P = P(a, b) = \frac{a - b}{2 \arcsin \frac{a-b}{a+b}},$$

A and G denote the arithmetic and geometric means of a and b, respectively.

Let $x = \arctan \frac{a-b}{a+b}$. Then $(\sin x)/x = T/Q$, $\cos x = A/Q$, and then inequalities (4.3) can be changed into

$$(4.6) \qquad M_\alpha \left(Q, A; \tfrac{2}{3} \right) < T < M_\beta \left(Q, A; \tfrac{2}{3} \right),$$

where T is the second Seiffert mean [15] *defined by*

$$T = T(a,b) = \frac{a-b}{2\arctan\frac{a-b}{a+b}},$$

Q *denotes the quadratic mean of a and b.*

Obviously, by Corollary 2, both the two double inequalities (4.5) (see [23]) and (4.6) hold if and only if $\alpha \le 4/5$ and $\beta \ge \frac{\ln 3 - \ln 2}{\ln \pi - \ln 2} \approx -0.898$, in which (4.6) seem to be new ones

In the same way, taking $k = 1$ in Theorem 3

Proposition 2. *For $x \in (0, \infty)$, inequality*

$$(4.7) \qquad \frac{2}{3}\left(\frac{\sinh x}{x}\right)^p + \frac{1}{3}\left(\frac{\tanh x}{x}\right)^p > 1$$

holds if and only if $p > 0$ or $p \le -\frac{4}{5}$.

Similar to Corollary 1, we have

Corollary 2. *Let $M_r(a, b; w)$ be defined by (4.2). Then for $x \in (0, \infty)$, the inequalities*

$$(4.8) \qquad M_\alpha\left(1, \cosh x; \tfrac{2}{3}\right) < \frac{\sinh x}{x} < M_\beta\left(1, \cosh x; \tfrac{2}{3}\right)$$

hold if and only if $\alpha \le 0$ and $\beta \ge 4/5$.

Remark 5. *Let $a > b > 0$ and let $x = \ln\sqrt{a/b}$. Then $(\sinh x)/x = L/G$, $\cosh x = A/G$, and then (4.8) can be changed into*

$$(4.9) \qquad M_\alpha\left(G, A; \tfrac{2}{3}\right) < L < M_\beta\left(G, A; \tfrac{2}{3}\right),$$

where L is the logarithmic means of a and b defined by

$$L = L(a,b) = \frac{a-b}{\ln a - \ln b}.$$

Making a change of variable $x = \operatorname{arcsinh}\frac{b-a}{a+b}$ yields $(\sinh x)/x = NS/A$, $\cosh x = Q/A$, where NS is the Nueman-Sándor mean defined by

$$NS = NS(a,b) = \frac{a-b}{2\operatorname{arcsinh}\frac{a-b}{a+b}}.$$

Thus, (4.8) is equivalent to

$$(4.10) \qquad M_\alpha\left(A, Q; \tfrac{2}{3}\right) < NS < M_\beta\left(A, Q; \tfrac{2}{3}\right).$$

Corollary 2 implies that the inequalities (4.9) and (4.10) hold if and only if $\alpha \le 0$ and $\beta \ge 4/5$. The second one in (4.10) is a new one.

Remark 6. *It should be pointed out that all inequalities involving $(\sin x)/x$ and $\cos x$ or $(\sinh x)/x$ and $\cosh x$ in this paper can be changed into the equivalent ones for means by variable substitutions mentioned previously. In what follows we no longer mention.*

4.2. **Wilker-Zhu type inequalities.** Letting $k = 2$ in Theorem 1 and 2, we have

Proposition 3. *For $x \in (0, \pi/2)$, inequality*

$$(4.11) \qquad \left(\frac{\sin x}{x}\right)^{2p} + \left(\frac{\tan x}{x}\right)^p > 2 > \left(\frac{\sin x}{x}\right)^{2q} + \left(\frac{\tan x}{x}\right)^q$$

holds if and only if $p > 0$ or $p \le -\frac{\ln 2}{2(\ln \pi - \ln 2)} \approx -0.767$ and $-3/5 \le q < 0$.

Note that

$$\frac{\left(\frac{\sin x}{x}\right)^{2p} + \left(\frac{\tan x}{x}\right)^p - 2}{\left(\frac{\sin x}{x}\right)^p + \frac{\sqrt{8 + \cos^{-2p} x} + \cos^{-p} x}{2}} = \left(\frac{x}{\sin x}\right)^{-p} - \frac{\sqrt{8 + \cos^{-2p} x} - \cos^{-p} x}{2},$$

by Proposition 3 the inequality

$$\frac{x}{\sin x} > \left(\frac{\sqrt{8 + \cos^{-2p} x} - \cos^{-p} x}{2}\right)^{-1/p}$$

or

$$\frac{\sin x}{x} < \left(\frac{\sqrt{8 + \cos^{-2p} x} + \cos^{-p} x}{4}\right)^{-1/p} := H_{-p}(\cos x)$$

holds for $x \in (0, \pi/2)$ if and only if $-p \ge \frac{\ln 2}{2(\ln \pi - \ln 2)}$, where H_r is defined on $(0, \infty)$ by

$$(4.12) \qquad H_r(t) = \left(\frac{\sqrt{8 + t^{2r}} + t^r}{4}\right)^{1/r} \text{ if } r \ne 0 \text{ and } H_0(t) = \sqrt[3]{t}.$$

Likewise, its reverse one holds if and only if $-p \le 3/5$. This result cab be stated as a corollary.

Corollary 3. *Let $H_r(t)$ be defined by (4.12). Then for $x \in (0, \pi/2)$, the inequalities*

$$(4.13) \qquad H_\alpha(\cos x) < \frac{\sin x}{x} < H_\beta(\cos x)$$

are true if and only if $\alpha \le 3/5$ and $\beta \ge \frac{\ln 2}{2(\ln \pi - \ln 2)} \approx 0.767$.

Taking $k = 2$ in Theorem 3, we have

Proposition 4. *For $x \in (0, \infty)$, the inequality*

$$\left(\frac{\sinh x}{x}\right)^{2p} + \left(\frac{\tanh x}{x}\right)^p > 2$$

holds if and only if $p > 0$ or $p \le -3/5$.

In a similar way, we get

Corollary 4. *Let $H_r(t)$ be defined by (4.12). Then for $x \in (0, \infty)$, the inequalities*

$$(4.14) \qquad H_\alpha(\cosh x) < \frac{\sinh x}{x} < H_\beta(\cosh x)$$

are true if and only if $\alpha \le 0$ and $\beta \ge 3/5$.

Now we give a generalization of inequalities (1.4) given by Zhu [26]

Proposition 5. *For fixed $k \geq 1$, both the chains of inequalities*

$$(4.15) \quad \frac{2}{k+2}\left(\frac{\sin x}{x}\right)^{kp} + \frac{k}{k+2}\left(\frac{\tan x}{x}\right)^{p} \geq \frac{k}{k+2}\left(\frac{\sin x}{x}\right)^{kp} + \frac{2}{k+2}\left(\frac{\tan x}{x}\right)^{p}$$
$$> \frac{2}{k+2}\left(\frac{x}{\sin x}\right)^{kp} + \frac{k}{k+2}\left(\frac{x}{\tan x}\right)^{p} > 1,$$

$$(4.16) \quad \frac{2}{k+2}\left(\frac{\sin x}{x}\right)^{kp} + \frac{k}{k+2}\left(\frac{\tan x}{x}\right)^{p} > \frac{2}{k+2}\left(\frac{x}{\tan x}\right)^{p} + \frac{k}{k+2}\left(\frac{x}{\sin x}\right)^{kp}$$
$$\geq \frac{2}{k+2}\left(\frac{x}{\sin x}\right)^{kp} + \frac{k}{k+2}\left(\frac{x}{\tan x}\right)^{p} > 1$$

hold for $x \in (0, \pi/2)$ if and only if $k \geq 2$ and $p \geq \frac{\ln(k+2)-\ln 2}{k(\ln \pi - \ln 2)}$.

Proof. The first inequality in (4.15) is equivalent to

$$\frac{2}{k+2}\left(\frac{\sin x}{x}\right)^{kp} + \frac{k}{k+2}\left(\frac{\tan x}{x}\right)^{p} - \frac{k}{k+2}\left(\frac{\sin x}{x}\right)^{kp} - \frac{2}{k+2}\left(\frac{\tan x}{x}\right)^{p}$$
$$= \frac{k-2}{k+2}\left(\left(\frac{\tan x}{x}\right)^{p} - \left(\frac{\sin x}{x}\right)^{kp}\right) > 0.$$

Due to $\frac{\tan x}{x} > 1$ and $\frac{\sin x}{x} < 1$, it holds for $x \in (0, \pi/2)$ if and only if

$$(k, p) \in \{k \geq 2, p > 0\} \cup \{1 \leq k \leq 2, p < 0\} := \Omega_1.$$

The second one is equivalent to

$$\frac{\frac{k}{k+2}\left(\frac{\sin x}{x}\right)^{kp} + \frac{2}{k+2}\left(\frac{\tan x}{x}\right)^{p}}{\frac{2}{k+2}\left(\frac{x}{\sin x}\right)^{kp} + \frac{k}{k+2}\left(\frac{x}{\tan x}\right)^{p}} > 1,$$

which can be simplified to

$$\left(\frac{\sin x}{x}\right)^{kp}\left(\frac{\tan x}{x}\right)^{p} = \left(\left(\frac{\sin x}{x}\right)^{k+1}\frac{1}{\cos x}\right)^{p} > 1.$$

It is true for $x \in (0, \pi/2)$ if and only if $(k, p) \in \{k+1 \geq 3, p \geq 0\} := \Omega_2$.

By Theorem 1, the third one in (4.15) holds for $x \in (0, \pi/2)$ if and only if

$$(k, p) \in \{k \geq 1, -p > 0\} \cup \{k \geq 1, -p \leq -\frac{\ln(k+2) - \ln 2}{k(\ln \pi - \ln 2)}\} := \Omega_3.$$

Hence, inequalities (4.15) hold for $x \in (0, \pi/2)$ if and only if

$$(k, p) \in \Omega_1 \cap \Omega_2 \cap \Omega_3 = \{k \geq 2, p \geq \frac{\ln(k+2) - \ln 2}{k(\ln \pi - \ln 2)}\},$$

which proves (4.15).

In the same way, we can prove (4.16), of which details are omitted. $\square$

Letting $k = 2$ in Proposition 5 we have

Corollary 5. *For $x \in (0, \pi/2)$, the inequalities (1.4) hold if and only if $p \geq \frac{\ln 2}{2(\ln \pi - \ln 2)} \approx 0.767$.*

Similarly, using Theorem 3 we easily prove the following

Proposition 6. *For fixed $k \geq 1$, the inequalities*

$$(4.17) \quad \frac{k}{k+2}\left(\frac{\sinh x}{x}\right)^{kp} + \frac{2}{k+2}\left(\frac{\tanh x}{x}\right)^{p} > \frac{2}{k+2}\left(\frac{x}{\sinh x}\right)^{kp} + \frac{k}{k+2}\left(\frac{x}{\tanh x}\right)^{p} > 1$$

hold $x \in (0, \infty)$ if and only if $k \geq 2$ and $p \geq \frac{12}{5(k+2)}$.

Letting $k = 2$ in Proposition 6 we have

Corollary 6. *For $x \in (0, \infty)$, the inequalities (1.5) hold if and only if $p \geq 3/5$.*

Remark 7. *Clearly, Corollaries 5 and 6 offer another method for solving the problems posed by Zhu in [28].*

4.3. **Other Wilker type inequalities.** Taking $k = 3, 4$ in Theorems 1 and 2, we obtain the following

Proposition 7. *For $x \in (0, \pi/2)$, inequality*

$$(4.18) \qquad \frac{2}{5}\left(\frac{\sin x}{x}\right)^{3p} + \frac{3}{5}\left(\frac{\tan x}{x}\right)^{p} > 1$$

holds if and only if $p > 0$ or $p \leq -\frac{\ln 5 - \ln 2}{3(\ln \pi - \ln 2)} \approx -0.676$. It is reversed if and only if $-12/25 \leq p < 0$.

Proposition 8. *For $x \in (0, \pi/2)$, inequality*

$$(4.19) \qquad \frac{1}{3}\left(\frac{\sin x}{x}\right)^{4p} + \frac{2}{3}\left(\frac{\tan x}{x}\right)^{p} > 1$$

holds if and only if $p > 0$ or $p \leq -\frac{\ln 3}{4(\ln \pi - \ln 2)} \approx -0.608$. It is reversed if and only if $-2/5 \leq p < 0$.

Putting $k = -3, -4$ in Theorem 3 we get

Proposition 9. *For $x \in (0, \infty)$, inequality*

$$(4.20) \qquad \left(\frac{\tanh x}{x}\right)^{p} < \frac{2}{3}\left(\frac{x}{\sinh x}\right)^{3p} + \frac{1}{3}$$

holds if and only if $p < 0$ or $p \geq 12/5$.

Proposition 10. *For $x \in (0, \pi/2)$, inequality*

$$(4.21) \qquad 2\left(\frac{\tanh x}{x}\right)^{p} < \left(\frac{x}{\sinh x}\right)^{4p} + 1$$

holds if and only if $p < 0$ or $p \geq 6/5$.

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POWER SUPPLY SERVICE CENTER, ZHEJIANG ELECTRIC POWER CORPORATION RESEARCH INSTITUTE, HANGZHOU CITY, ZHEJIANG PROVINCE, 310009, CHINA

E-mail address: yzhkm@163.com